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A variational Barban–Davenport–Halberstam Theorem

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ABSTRACT

We prove variational forms of the Barban–Davenport–Halberstam Theorem and the large sieve inequality. We apply our result to prove an estimate for the sum of the squares of prime differences, averaged over arithmetic progressions.

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1. Introduction

The prime number theorem implies the asymptotic $\psi(x) \sim x$, while the Riemann hypothesis predicts a bound of $|\psi(x) - x| \ll_{\epsilon} x^{\frac{1}{2} + \epsilon}$ on the error term. This extends naturally to arithmetic progressions, where the asymptotic $\psi(x; q, a) \sim \frac{x}{\phi(q)}$ holds for all coprime a and q . We recall that

$$\psi(x; q, a) := \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n).$$

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Under the generalized Riemann hypothesis, one obtains the error bound $|\psi(x; q, a) - \frac{x}{\phi(q)}| \ll_\epsilon x^{\frac{1}{2}+\epsilon}$. The stronger bound of $\ll_\epsilon x^{\frac{1}{2}+\epsilon} \phi(q)^{-\frac{1}{2}}$ is also conjectured. (For further definitions, see Section 2. For background material, see [2].)

An unconditional bound on the averaged error term for this is provided by the Barban–Davenport–Halberstam Theorem [2], which states:

Theorem 1. (Barban–Davenport–Halberstam) Let $A > 0$. For all positive real numbers x and Q satisfying $x(\log(x))^{-A} \leq Q \leq x$,

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a, q) = 1}} \left(\psi(x; q, a) - \frac{x}{\phi(q)} \right)^2 \ll_A xQ \log(x).$$

We note that this holds also for the quantity $\theta(x; q, a)$, since the differences between $\psi(x; q, a)$ and $\theta(x; q, a)$ are of lower order.

An even stronger bound is due to Montgomery [10], refining work of Uchiyama [15] (see also the refinement of Hooley [6]):

Theorem 2. Let $A > 0$. For all positive real numbers x and Q satisfying $x(\log(x))^{-A} \leq Q \leq x$,

$$\sum_{q \leq Q} \max_{y \leq x} \sum_{\substack{a \leq q \\ (a, q) = 1}} \left(\psi(y; q, a) - \frac{y}{\phi(q)} \right)^2 \ll_A xQ \log(x).$$

We note that the quantity on the left has potentially increased compared to the quantity in Theorem 1, while the bound on the right is the same, up to the implicit constant.

Another variant of Theorem 1 is due to Uchiyama [15]:

Theorem 3. Let $A > 0$. For all positive real numbers x and Q satisfying $x(\log(x))^{-A} \leq Q \leq x$,

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a, q) = 1}} \max_{y \leq x} \left(\psi(y; q, a) - \frac{y}{\phi(q)} \right)^2 \ll_A xQ \log^3(x).$$

This is incomparable to Theorems 1 and 2, since the quantity being bounded is larger and the bound obtained is worse. Hooley, in [7], has announced a refinement to the $\log^3(N)$ for certain values of Q . This seems to have not yet appeared, however.

We work with the function θ instead of ψ because it is more convenient for our purposes, though this is a minor difference. To further refine our understanding of the deviation of $\theta(x; q, a)$ from its average value of $\frac{x}{\phi(q)}$, we introduce a variational operator in place of the maximal one in Theorem 3. Letting $\{c_n\}_{n=1}^N$ be a finite sequence of complex numbers and letting \mathcal{P}_N denote the set of partitions of $[N] := \{1, 2, \dots, N\}$ into disjoint intervals, we define the r -variation of the sequence to be:

$$\|\{c_n\}_{n=1}^N\|_{V^r} := \max_{\pi \in \mathcal{P}_N} \left(\sum_{I \in \pi} \left| \sum_{n \in I} c_n \right|^r \right)^{\frac{1}{r}}.$$

We can think of $\theta(x; q, a)$ as a sum over a sequence $\{b_n\}_{n=1}^N$, where $N = \lfloor x \rfloor$ and

$$b_n := \begin{cases} \log(n), & n \text{ prime;} \\ 0, & \text{otherwise.} \end{cases}$$

For an interval I , we define

$$\theta(I; q, a) := \sum_{n \in I} b_n.$$

Letting $|I|$ denote the number of integers contained in I , we then have that

$$\max_{\pi \in \mathcal{P}_N} \sum_{I \in \pi} \left(\theta(I; q, a) - \frac{|I|}{\phi(q)} \right)^2$$

is the square of the 2-variation of the sequence $\{b_n - 1/\phi(q)\}_{n=1}^N$.

Our main result is an upper bound on this quantity, summing over $q \leq Q$ and a coprime to q as in the above theorems. This is a strengthening of Theorem 3, since we obtain the same bound (up to the implied constant) on a larger quantity. To simplify our notation, we let \mathcal{P}_x denote the set of partitions of $\{1, \dots, \lfloor x \rfloor\}$ into disjoint intervals. We prove:

Theorem 4. *Let $A > 0$. For all positive real numbers x and Q satisfying $x(\log(x))^{-A} \leq Q \leq x$,*

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a, q) = 1}} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \left(\theta(I; q, a) - \frac{|I|}{\phi(q)} \right)^2 \ll_A xQ \log^3(x).$$

We also establish a variant of this, obtaining a better bound by allowing the partition to depend only on q and not on a :

Theorem 5. *Let $A > 0$. For all positive real numbers x and Q satisfying $x(\log(x))^{-A} \leq Q \leq x$,*

$$\sum_{q \leq Q} \max_{\pi \in \mathcal{P}_x} \sum_{\substack{a \leq q \\ (a, q) = 1}} \sum_{I \in \pi} \left(\theta(I; a, q) - \frac{|I|}{\phi(q)} \right)^2 \ll_A xQ \log^2(x).$$

In comparison to Theorem 2, this maximizes over partitions instead of restricting to partial sums, but the bound obtained is worse by a multiplicative $\log(x)$ factor.

The introduction of this maximum over partitions allows us to apply our theorem to prove a weakened, averaged version of a conjecture made by Erdős. We let p_i denote the i -th prime. Erdős made the following conjecture:

Conjecture 6. (See Erdős [3].)

$$\sum_{p_{i+1} \leq x} (p_{i+1} - p_i)^2 \ll x \log(x).$$

This asymptotic is heuristically suggested by the prime number theorem, which implies the reverse inequality. Assuming the Riemann hypothesis, Selberg [14] obtained the bound

$$\sum_{p_{i+1} \leq x} (p_{i+1} - p_i)^2 \ll x \log^3(x).$$

It is natural to extend the conjecture to arithmetic progressions. Fixing a, q such that $(a, q) = 1$, we let $p_i^{a, q}$ denote the i -th prime congruent to a modulo q . One then formulates the conjecture as:

Conjecture 7. For any a, q such that $(a, q) = 1$,

$$\sum_{p_i^{a,q} \leq x} \left(\frac{p_{i+1}^{a,q} - p_i^{a,q}}{\phi(q)} \right)^2 \ll \frac{x \log(x)}{\phi(q)}.$$

If we then sum over all $q \leq Q$ and all a coprime to q , we would expect to get $\ll Qx \log(x)$. We derive the following weaker bound:

Corollary 8. Let $A > 0$. For all positive real numbers x and Q satisfying $x(\log(x))^{-A} \leq Q \leq x$,

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a,q)=1}} \sum_{p_{i+1}^{a,q} \leq x} \left(\frac{p_{i+1}^{a,q} - p_i^{a,q}}{\phi(q)} \right)^2 \ll Qx \log^3(x).$$

This can be viewed as an averaged, unconditional version of Selberg's bound, and is easily obtained from Theorem 4.

More generally, the study of variational quantities introduces new and interesting questions. For example, is there an elementary function f such that

$$\max_{\pi \in \mathcal{P}_x} \sum_{l \in \pi} \left(\sum_{n \in l} (\Lambda(n) - 1) \right)^2 \sim f(x)? \quad (1)$$

The prime number theorem gives an (asymptotic) lower bound of $x \log(x)$ on this quantity. We note, however, that one cannot hope to have $f(x) = x \log(x)$. This follows from the work of Cheer and Goldston [1], who proved

Theorem 9. For any $\epsilon > 0$, there exists an X_0 such that for all $x > X_0$,

$$\sum_{p_{i+1} \leq x} |p_{i+1} - p_i|^2 \geq (193/192 - \epsilon)x \log x.$$

(Note, as seen from Lemma 10 below, the contribution to (1) from prime powers is of lower order.) This does not rule out the possibility of $f(x) = Cx \log(x)$ for some larger C , for example.

2. Preliminaries

We first recall some standard definitions. When q is a positive integer, $\phi(q)$ denotes the Euler totient function. For positive integers a and q , (a, q) denotes the g.c.d. of a and q .

For a positive real number x , we define

$$\begin{aligned} \psi(x) &= \sum_{n \leq x} \Lambda(n) = \sum_{p^\alpha \leq x} \log(p), \\ \theta(x) &= \sum_{p \leq x} \log(p). \end{aligned}$$

Here, \log denotes the natural logarithm. The latter sum for ψ is over prime powers p^α , while the sum for θ is over primes p . $\Lambda(n)$ denotes the von Mangoldt function, which is equal to $\log(p)$ whenever n is a power of a prime p and equal to 0 otherwise.

Letting a and q be positive integers, we similarly define

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n), \quad \theta(x; q, a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log(p).$$

Letting I be an interval, we also define

$$\psi(I; q, a) = \sum_{\substack{n \in I \\ n \equiv a \pmod{q}}} \Lambda(n), \quad \theta(I; q, a) = \sum_{\substack{p \in I \\ p \equiv a \pmod{q}}} \log(p).$$

The size of an interval I is defined to be the number of integers it contains, and is denoted by $|I|$. For a fixed positive real number x , we let \mathcal{P}_x denote the set of all partitions of $[1, \lfloor x \rfloor]$ into intervals. Thus an element $\pi \in \mathcal{P}_x$ is a collection of disjoint intervals whose union is the interval from 1 to $\lfloor x \rfloor$.

We recall the prime number theorem, which states that $\psi(x) \sim x$. We will later also use the following standard fact (we include the short proof here for completeness):

Lemma 10. For $x \geq 2$, $\theta(x) = \psi(x) + O(x^{\frac{1}{2}})$.

Proof. Since $p^\alpha \leq x$ holds if and only if $p \leq x^{\frac{1}{\alpha}}$, we have

$$\psi(x) = \sum_{\alpha=1}^{\infty} \theta(x^{\frac{1}{\alpha}}) \quad \text{and} \quad \psi(x) - \theta(x) = \sum_{\alpha \geq 2} \theta(x^{\frac{1}{\alpha}}).$$

Noting that $x^{\frac{1}{\alpha}} \geq 2$ only for $\alpha = O(\log x)$ and $\theta(x^{\frac{1}{\alpha}}) \leq \psi(x^{\frac{1}{\alpha}}) \ll x^{\frac{1}{\alpha}}$, we see this is $\ll x^{\frac{1}{2}} + x^{\frac{1}{3}} \log x \ll x^{\frac{1}{2}}$. \square

3. A variational form of the Barban–Davenport–Halberstam Theorem

We now prove:

Theorem 4. Let $A > 0$. For all positive real numbers x and Q satisfying $x(\log(x))^{-A} \leq Q \leq x$,

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a, q) = 1}} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \left(\theta(I; q, a) - \frac{|I|}{\phi(q)} \right)^2 \ll_A x Q \log^3(x).$$

We will deduce this by combining the proof of the standard Barban–Davenport–Halberstam Theorem with some combinatorial arguments and a variational form of the Siegel–Walfisz Theorem that is developed in the following subsection.

For a fixed positive integer q , we consider Dirichlet characters modulo q . A function $\chi : \mathbb{Z}_q^* \rightarrow \mathbb{C}$ is called a Dirichlet character modulo q if it is a group homomorphism. We can extend such a χ to be a function from \mathbb{Z} to \mathbb{C} by defining $\chi(n)$ to be equal to the value of the character on the residue class of n modulo q when n is coprime to q and 0 otherwise. From now on, we will consider Dirichlet characters to be functions on \mathbb{Z} . A character $\chi \pmod{q}$ is said to be *primitive* if its period as a function on \mathbb{Z} is precisely q (conversely it is non-primitive if it has a smaller period dividing q).

We fix positive integers M and N . Given a Dirichlet character $\chi \bmod q$ and complex numbers $\{a_n\}_{n=M+1}^{M+N}$, we define

$$T(\chi) = \sum_{n=M+1}^{M+N} a_n \chi(n).$$

More generally, for any interval $I \subseteq [M+1, M+N]$, we define

$$T(\chi, I) = \sum_{n \in I} a_n \chi(n).$$

The large sieve inequality [2] states:

Theorem 11 (*The large sieve inequality*). For any positive integers Q, M, N and complex numbers $\{a_n\}_{n=M+1}^{M+N}$:

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* |T(\chi)|^2 \ll (N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2.$$

Here, the inner sum \sum_{χ}^* is over the **primitive** characters modulo q (this is what the $*$ superscript signifies).

In our proof of Theorem 4, we will use the large sieve inequality directly as it is stated above. However, we will later establish variational versions of this in Sections 4 and 5.

3.1. A variational form of the Siegel–Walfisz Theorem

For a positive real number x and a Dirichlet character $\chi \bmod q$, we define

$$\psi(x, \chi) = \sum_{p^\alpha \leq x} \chi(p^\alpha) \log(p) \quad \text{and} \quad \theta(x, \chi) = \sum_{p \leq x} \chi(p) \log(p).$$

For an interval I , we similarly define

$$\theta(I, \chi) = \sum_{p \in I} \chi(p) \log(p).$$

We refer to the unique $\chi \bmod q$ that takes the value 1 on all integers coprime to q as the *principal* character modulo q , and all other characters as non-principal.

The Siegel–Walfisz Theorem [2] states:

Theorem 12 (*Siegel–Walfisz Theorem*). Let A be a positive real number. Then there exists some positive constant c_A depending only on A such that

$$|\psi(x, \chi)| \ll_A x e^{-c_A \log^{\frac{1}{2}}(x)}$$

for all non-principal characters $\chi \bmod q$ for all moduli $q \leq \log^A(x)$.

We will find it more convenient to work with the following corollary:

Corollary 13. *Let A be a positive real number. Then there exists some positive constant c_A depending only on A such that*

$$|\theta(x, \chi)| \ll_A x e^{-c_A \log^{\frac{1}{2}}(x)}$$

for all non-principal characters $\chi \bmod q$ for all moduli $q \leq \log^A(x)$.

Proof. By the triangle inequality, $|\theta(x, \chi)| \leq |\psi(x, \chi)| + |\psi(x, \chi) - \theta(x, \chi)|$. The first quantity is bounded by Theorem 12. To bound the second quantity, we observe

$$|\psi(x, \chi) - \theta(x, \chi)| = \left| \sum_{\substack{p^\alpha \leq x \\ \alpha > 1}} \chi(p^\alpha) \log(p) \right| \leq \sum_{\substack{p^\alpha \leq x \\ \alpha > 1}} \log(p) = \psi(x) - \theta(x),$$

by the triangle inequality and the fact that $|\chi(p^\alpha)|$ is always either 0 or 1. Applying Lemma 10, we see that $|\psi(x, \chi) - \theta(x, \chi)| \ll x^{\frac{1}{2}}$, where the implicit constant is independent of q and χ . \square

We now prove a variational form of this:

Lemma 14. *Let A be a positive real number. Then there exists some positive constant c'_A depending only on A such that*

$$\sqrt{\max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} |\theta(I, \chi)|^2} \ll_A x e^{-c'_A \log^{\frac{1}{2}}(x)} \quad (2)$$

for all non-principal characters $\chi \bmod q$ for all moduli $q \leq \log^A(x)$.

Proof. Since every $I \in \pi$ is a subinterval of $[1, x]$, the left-hand side of (2) is

$$\ll \sqrt{\max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} |\theta(I, \chi)| \cdot \max_{J \subseteq [1, x]} |\theta(J, \chi)|}. \quad (3)$$

We consider the inner quantity $\sum_{I \in \pi} |\theta(I, \chi)|$. By definition, we have

$$\sum_{I \in \pi} |\theta(I, \chi)| = \sum_{I \in \pi} \left| \sum_{p \in I} \chi(p) \log(p) \right|.$$

Applying the triangle inequality, this is

$$\leq \sum_{I \in \pi} \sum_{p \in I} |\chi(p) \log(p)| \leq \sum_{I \in \pi} \sum_{p \in I} \log(p) = \theta(x).$$

Here, we have used the fact that $|\chi(p)|$ is always either 1 or 0.

We then have that (3) is

$$\ll \sqrt{\theta(x) \max_{J \subseteq [1, x]} |\theta(J, \chi)|}.$$

Since $\theta(x) \leq \psi(x) \ll x$, this is

$$\ll \sqrt{x \max_{J \subseteq [1, x]} |\theta(J, \chi)|}. \quad (4)$$

We consider the quantity $\max_{J \subseteq [1, x]} |\theta(J, \chi)|$. We observe that this is $\ll \max_{y \leq x} |\theta(y, \chi)|$. We will upper bound this quantity for each y separately. For larger y values, we will employ Corollary 13 for the value $2A$ (using $2A$ instead of A will allow us to apply the corollary to a larger range of y values). We let c_{2A} denote the constant for $2A$ in the exponent. More precisely, for y such that $q \leq \log^{2A}(y)$, we have $|\theta(y, \chi)| \ll_A y e^{-c_{2A} \log^{\frac{1}{2}}(y)}$ by Corollary 13. Since $y \leq x$, this is $\ll_A x e^{-c_{2A} \log^{\frac{1}{2}}(x)}$.

We now consider y such that $\log^{2A}(y) \leq q$. This is equivalent to the condition $y \leq e^{q^{\frac{1}{2A}}}$. For these small y values, we will use the basic estimate $|\theta(y, \chi)| \ll \theta(y) \ll y$. Since $\log^A(x) \geq q$ holds by assumption, we have $\log(x) \geq q^{\frac{1}{A}} \geq \log^2(y)$. We then have $y \leq e^{\log^{\frac{1}{2}}(x)} \ll_A x e^{-c_{2A} \log^{\frac{1}{2}}(x)}$. Hence,

$$\max_{y \leq x} |\theta(y, \chi)| \ll_A x e^{-c_{2A} \log^{\frac{1}{2}}(x)}.$$

Thus, the quantity in (4) is

$$\ll_A \sqrt{x^2 e^{-c_{2A} \log^{\frac{1}{2}}(x)}} = x e^{-\frac{1}{2} \cdot c_{2A} \log^{\frac{1}{2}}(x)}.$$

This proves Lemma 14 with $c'_A := \frac{1}{2} \cdot c_{2A}$. \square

We note that, conditional on the generalized Riemann hypothesis, for a non-principal Dirichlet of modulus q one has the bound

$$|\psi(x, \chi)| \ll x^{1/2} \log(x) \log(qx)$$

where the implied constant is absolute (see Theorem 13.7 in [12]). This can be used as in the argument above to obtain:

Lemma 15. *Let χ be a non-principal character mod q . Assuming the generalized Riemann hypothesis, we have that*

$$\max_{\pi \in \mathcal{P}_\chi} \sum_{I \in \pi} |\theta(I, \chi)|^2 \ll x^{3/2} \log(x) \log(qx). \quad (5)$$

This could be used in place of Lemma 14 in the following arguments to conditionally extend the range of Q in the statements of Theorems 4 and 5. This is quite routine, and we omit the details. It may be possible to further improve (conditionally) the exponent of the $x^{3/2}$ term. We leave this as an interesting open problem.

3.2. Proof of Theorem 4

We now bound the quantity

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a,q)=1}} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \left(\theta(I; q, a) - \frac{|I|}{\phi(q)} \right)^2. \quad (6)$$

The structure of our proof will resemble the proof of the non-variational version of the theorem in [2].

We let 2^k denote the smallest power of two that is $\geq x$. We can then decompose $[1, 2^k]$ into dyadic intervals $I_{c,\ell} = ((c-1)2^\ell, c2^\ell]$, where ℓ ranges from 0 to 2^k and c ranges from 1 to $2^{k-\ell}$. We note the following lemma [8]:

Lemma 16. Any subinterval of $S \subset [1, 2^k]$ can be expressed as the disjoint union of intervals of the form $I_{c,\ell}$, such as

$$S = \bigcup_m I_{c_m, \ell_m}$$

where at most two of the intervals I_{c_m, ℓ_m} in the union are of each size, and where the union consists of at most $2k$ intervals.

In other words, each $I \subseteq [x]$ can be decomposed as a disjoint union of these dyadic intervals using at most two intervals on each level ℓ . We let $D(I)$ denote the set of dyadic intervals in the decomposition of I . We observe

$$\theta(I; q, a) - \frac{|I|}{\phi(q)} = \sum_{J \in D(I)} \left(\theta(J; q, a) - \frac{|J|}{\phi(q)} \right)$$

for any I , since $\sum_{J \in D(I)} |J| = |I|$. For each ℓ , we let $D_\ell(I)$ denote the intervals in $D(I)$ on level ℓ (so $|D_\ell(I)| \leq 2$). We can rewrite (6) as:

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a,q)=1}} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \left(\sum_{\ell=0}^{2^k} \sum_{J \in D_\ell(I)} \theta(J; q, a) - \frac{|I|}{\phi(q)} \right)^2.$$

By the triangle inequality for the ℓ^2 norm, we have

$$\begin{aligned} & \sqrt{\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a,q)=1}} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \left(\theta(I; q, a) - \frac{|I|}{\phi(q)} \right)^2} \\ & \ll \sum_{\ell=0}^k \left(\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a,q)=1}} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \left(\sum_{J \in D_\ell(I)} \theta(J; q, a) - \frac{|J|}{\phi(q)} \right)^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (7)$$

Since $|D_\ell(I)| \leq 2$ for all ℓ , I and each dyadic interval can appear in $D(I)$ for at most one $I \in \pi$,

$$\max_{\pi \in \mathcal{P}_X} \sum_{I \in \pi} \left(\sum_{J \in D_\ell(I)} \theta(J; q, a) - \frac{|J|}{\phi(q)} \right)^2 \ll \sum_{c=1}^{2^{k-\ell}} \left(\theta(I_{c,\ell}; q, a) - \frac{|I_{c,\ell}|}{\phi(q)} \right)^2$$

for all a, q, ℓ . Therefore the quantity in (7) is

$$\ll \sum_{\ell=0}^k \left(\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a,q)=1}} \sum_{c=1}^{2^{k-\ell}} \left(\theta(I_{c,\ell}; q, a) - \frac{|I_{c,\ell}|}{\phi(q)} \right)^2 \right)^{\frac{1}{2}}. \quad (8)$$

For each fixed ℓ , we consider the quantity

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a,q)=1}} \sum_{c=1}^{2^{k-\ell}} \left(\theta(I_{c,\ell}; q, a) - \frac{|I_{c,\ell}|}{\phi(q)} \right)^2. \quad (9)$$

We will bound this quantity using the large sieve inequality and the Siegel–Walfisz Theorem, so we need to first express it in terms of characters χ modulo q . For this, we will introduce some convenient notation. Fixing q , we let χ_0 denote the principal character modulo q . For any character χ modulo q , we define

$$\theta'(I, \chi) := \begin{cases} \theta(I, \chi), & \chi \neq \chi_0; \\ \theta(I, \chi_0) - |I|, & \chi = \chi_0. \end{cases}$$

We will employ the following lemma:

Lemma 17. *For any interval I and any coprime positive integers q, a ,*

$$\theta(I; q, a) - \frac{|I|}{\phi(q)} = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \theta'(I, \chi),$$

where $\bar{\chi}$ denotes the character obtained from χ by complex conjugation.

Proof. We note that for each integer n ,

$$\sum_{\chi \bmod q} \chi(n) = \begin{cases} \phi(q), & n \equiv 1 \pmod{q}; \\ 0, & \text{otherwise.} \end{cases}$$

For any integer a coprime to q , we let \bar{a} denote an integer such that $\bar{a}a \equiv 1 \pmod{q}$. Then, for any $\chi \bmod q$, $\bar{\chi}(a)\chi(n) = \chi(\bar{a})\chi(n) = \chi(\bar{a}n)$. Since $\bar{a}n \equiv 1 \pmod{q}$ if and only if $n \equiv a \pmod{q}$, we have

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a)\chi(n) = \begin{cases} 1, & n \equiv a \pmod{q}; \\ 0, & \text{otherwise.} \end{cases}$$

We then observe

$$\begin{aligned}\theta(I; q, a) &= \sum_{\substack{p \in I \\ p \equiv a \pmod{q}}} \log(p) \\ &= \frac{1}{\phi(q)} \sum_{p \in I} \log(p) \sum_{\chi \pmod{q}} \bar{\chi}(a) \chi(p) \\ &= \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \theta(I, \chi).\end{aligned}$$

By definition of $\theta'(I, \chi)$, it then follows that

$$\theta(I; q, a) - \frac{|I|}{\phi(q)} = \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \theta'(I, \chi). \quad \square$$

The quantity (9) can then be expressed as

$$\sum_{q \leq Q} \frac{1}{\phi(q)^2} \sum_{c=1}^{2^{k-\ell}} \sum_{\substack{a \leq q \\ (a, q)=1}} \left| \sum_{\chi \pmod{q}} \bar{\chi}(a) \theta'(I_{c, \ell}, \chi) \right|^2. \quad (10)$$

For fixed q and c , the inner quantity can be expanded as:

$$= \sum_{\substack{a \leq q \\ (a, q)=1}} \sum_{\chi_1 \pmod{q}} \sum_{\chi_2 \pmod{q}} \bar{\chi}_1(a) \chi_2(a) \theta'(I_{c, \ell}, \chi_1) \overline{\theta'(I_{c, \ell}, \chi_2)}.$$

Reordering the sums, this is

$$\sum_{\chi_1 \pmod{q}} \sum_{\chi_2 \pmod{q}} \theta'(I_{c, \ell}, \chi_1) \overline{\theta'(I_{c, \ell}, \chi_2)} \sum_{\substack{a \leq q \\ (a, q)=1}} \bar{\chi}_1(a) \chi_2(a).$$

The innermost sum is now the inner product of the characters χ_1, χ_2 . Since the distinct characters modulo q are orthogonal under this inner product, this innermost sum is 0 unless $\chi_1 = \chi_2$. Therefore, (10) is equal to

$$\sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{c=1}^{2^{k-\ell}} \sum_{\chi \pmod{q}} |\theta'(I_{c, \ell}, \chi)|^2. \quad (11)$$

In order to use the large sieve inequality as stated in Theorem 11, we need to adjust our character sum to be over the primitive characters modulo q instead of all characters modulo q . For this, we first note that every character χ modulo q is induced by some primitive character χ_1 modulo q_1 where $q_1 \leq q$. We then have:

Lemma 18. For any I and any character χ modulo q induced by χ_1 modulo q_1 ,

$$|\theta'(I, \chi_1) - \theta'(I, \chi)| \leq \log(q).$$

Proof. For all integers n coprime to q , $\chi(n) = \chi_1(n)$. In fact,

$$\theta'(I, \chi_1) - \theta'(I, \chi) = \sum_{\substack{p \in I \\ p|q}} \chi_1(p) \log(p).$$

Therefore,

$$|\theta'(I, \chi_1) - \theta'(I, \chi)| \leq \sum_{\substack{p \in I \\ p|q}} \log(p) \leq \log(q).$$

To see the final inequality, consider the prime factorization of $q = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Then $\log(q) = \alpha_1 \log(p_1) + \cdots + \alpha_r \log(p_r)$. \square

As a consequence of Lemma 18, we have $|\theta'(I_{c,\ell}, \chi)|^2 \ll |\theta'(I_{c,\ell}, \chi_1)|^2 + \log^2 q$ for all dyadic intervals $I_{c,\ell}$ and all non-primitive characters χ . Thus, the quantity in (11) is

$$\begin{aligned} &\ll \sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{c=1}^{2^{k-\ell}} \sum_{\chi \bmod q} (\log^2(q) + |\theta'(I_{c,\ell}, \chi_1)|^2) \\ &= \sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{c=1}^{2^{k-\ell}} \sum_{\chi \bmod q} \log^2(q) + \sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{c=1}^{2^{k-\ell}} \sum_{\chi \bmod q} |\theta'(I_{c,\ell}, \chi_1)|^2. \end{aligned} \quad (12)$$

As above, χ_1 here denotes the primitive character that induces χ .

We bound the contribution of this first sum to the quantity in (8) (noting that there are $\phi(q)$ characters modulo q):

$$\sum_{0 \leq \ell \leq k} \sqrt{\sum_{q \leq Q} 2^{k-\ell} \log^2(q)} = \left(\sum_{0 \leq \ell \leq k} (2^{\frac{1}{2}})^{k-\ell} \right) \sqrt{\sum_{q \leq Q} \log^2(q)} \ll 2^{\frac{k}{2}} \sqrt{Q \log^2(Q)}.$$

Since (8) is an upper bound on the square root of (6), the contribution to (6) is therefore $\ll 2^k Q \log^2(Q) \ll x Q \log^2(x)$, which is acceptable.

It thus suffices to consider

$$\sum_{q \leq Q} \frac{1}{\phi(q)} \sum_{c=1}^{2^{k-\ell}} \sum_{\chi \bmod q} |\theta'(I_{c,\ell}, \chi_1)|^2 \quad (13)$$

for each fixed ℓ . Each primitive character χ_1 modulo q_1 induces characters χ modulo q for every q that is a multiple of q_1 . We can use this to rewrite the above quantity in terms of a sum over only primitive characters:

$$= \sum_{q \leq Q} \sum_{c=1}^{2^{k-\ell}} \sum_{\chi \bmod q}^* |\theta'(I_{c,\ell}, \chi)|^2 \left(\sum_{j \leq \frac{Q}{q}} \frac{1}{\phi(jq)} \right).$$

We note that

$$\sum_{j \leq \frac{Q}{q}} \frac{1}{\phi(jq)} \ll \phi(q)^{-1} \log(2Q/q)$$

(see [2, p. 163]), so this is

$$\ll \sum_{q \leq Q} \frac{1}{\phi(q)} \log\left(\frac{2Q}{q}\right) \sum_{c=1}^{2^{k-\ell}} \sum_{\chi \bmod q}^* |\theta'(I_{c,\ell}, \chi)|^2. \quad (14)$$

We will split this sum over $q \leq Q$ into ranges and bound each piece separately. For a fixed $1 \leq U \leq Q$, we consider

$$\sum_{c=1}^{2^{k-\ell}} \sum_{U < q \leq 2U} \frac{1}{\phi(q)} \log\left(\frac{2Q}{q}\right) \sum_{\chi \bmod q}^* |\theta(I_{c,\ell}, \chi)|^2. \quad (15)$$

Note that we have switched notation from θ' to θ here without changing the quantity, since θ' and θ only differ on the trivial character, and this is only included in the primitive characters modulo q when $q = 1$. Since $U \geq 1$ and our sum here is over $q > U$, θ and θ' behave identically here.

We observe that for each fixed c , the contribution to (15) is

$$\ll U^{-1} \log\left(\frac{2Q}{U}\right) \sum_{q \leq 2U} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* |\theta(I_{c,\ell}, \chi)|^2.$$

Letting $a_p := \log(p)$ for primes p and $a_n := 0$ for all non-primes n , we apply Theorem 11 to see that this is

$$\ll U^{-1} \log\left(\frac{2Q}{U}\right) (|I_{c,\ell}| + U^2) \sum_{p \in I_{c,\ell}} \log^2(p).$$

We define $Q_1 := \log^{A+1}(x)$. We consider the values $U = Q2^{-j}$ as j ranges from 0 to $J := \lceil \log(\frac{Q}{Q_1}) \rceil$. We then have:

$$\sum_{Q_1 < q \leq Q} \frac{1}{\phi(q)} \log\left(\frac{2Q}{q}\right) \sum_{\chi \bmod q}^* |\theta(I_{c,\ell}, \chi)|^2 \ll Q^{-1} \sum_{j=0}^J 2^j (j+1) (|I_{c,\ell}| + Q^2 2^{-2j}) \sum_{p \in I_{c,\ell}} \log^2(p).$$

We expand the latter quantity as

$$= |I_{c,\ell}| Q^{-1} \left(\sum_{j=0}^J (j+1) 2^j \right) \left(\sum_{p \in I_{c,\ell}} \log^2(p) \right) + Q \left(\sum_{j=0}^J (j+1) 2^{-j} \right) \left(\sum_{p \in I_{c,\ell}} \log^2(p) \right).$$

Inserting this into the sum over the c values, we then have

$$\begin{aligned} & \sum_{c=1}^{2^{k-\ell}} \sum_{Q_1 < q \leq Q} \frac{1}{\phi(q)} \log\left(\frac{2Q}{q}\right) \sum_{\chi \bmod q}^* |\theta(I_{c,\ell}, \chi)|^2 \\ & \ll \sum_{c=1}^{2^{k-\ell}} |I_{c,\ell}| Q^{-1} \left(\sum_{j=0}^J (j+1) 2^j \right) \left(\sum_{p \in I_{c,\ell}} \log^2(p) \right) + \sum_{c=1}^{2^{k-\ell}} Q \left(\sum_{j=0}^J j 2^{-j} \right) \left(\sum_{p \in I_{c,\ell}} \log^2(p) \right). \end{aligned}$$

We observe that

$$\sum_{c=1}^{2^{k-\ell}} \sum_{p \in I_{c,\ell}} \log^2(p) = \sum_{p \leq 2^k} \log^2(p),$$

since the union of the intervals $I_{c,\ell}$ as c ranges from 1 to $2^{k-\ell}$ is equal to the interval $[2^k]$. We then have that $\sum_{p \leq 2^k} \log^2(p) \ll k 2^k$. We also note that

$$\sum_{j=0}^J (j+1) 2^j = J 2^{J+1} + 1 \ll J 2^J \quad \text{and} \quad \sum_{j=0}^J (j+1) 2^{-j} \ll 1.$$

Therefore, we have shown

$$\sum_{c=1}^{2^{k-\ell}} \sum_{Q_1 < q \leq Q} \frac{1}{\phi(q)} \log\left(\frac{2Q}{q}\right) \sum_{\chi \bmod q}^* |\theta(I_{c,\ell}, \chi)|^2 \ll Q^{-1} (2^\ell) (J 2^J) (k 2^k) + Q (k 2^k). \quad (16)$$

Recalling that $2^k \ll x$, $k \ll \log(x)$, $J = \lceil \log(\frac{Q}{Q_1}) \rceil$, and $Q \leq x$, we see that the contribution to (14) from q 's between Q_1 and Q is

$$\ll Q_1^{-1} (2^\ell) (x \log^2(x)) + Q (x \log(x)).$$

We now consider values of $q \leq Q_1$. For every primitive character χ modulo q where $q > 1$, χ is non-principal. Note that for the principal character χ_0 modulo 1, $\theta'(I_{c,\ell}, \chi_0) = \theta(I_{c,\ell}, \chi_0) - |I_{c,\ell}| = 0$. Thus, the contribution to (14) from values $q \leq Q_1$ can be written as:

$$\sum_{1 < q \leq Q_1} \frac{1}{\phi(q)} \log\left(\frac{2Q}{q}\right) \sum_{\chi \bmod q}^* \sum_{c=1}^{2^{k-\ell}} |\theta'(I_{c,\ell}, \chi)|^2. \quad (17)$$

This innermost sum over c is a sum over a partition of $[2^k]$, so we can apply Lemma 14 (with $A+1$ as the constant) to conclude that

$$\sum_{c=1}^{2^{k-\ell}} |\theta'(I_{c,\ell}, \chi)|^2 \ll_A x^2 e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)}$$

for some positive constant \tilde{c}_A depending only on A . The quantity in (17) is then

$$\ll_A Q_1 \log(Q) \cdot x^2 e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)}.$$

Putting this all together, we have that the quantity in (14) is:

$$\ll_A Q_1 \log(Q) \cdot x^2 e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)} + Q_1^{-1}(2^\ell)(x \log^2(x)) + Q(x \log(x)).$$

Thus, the contribution to (8) is bounded by

$$\begin{aligned} &\ll_A \sum_{\ell=0}^k \sqrt{Q_1 \log(Q) \cdot x^2 e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)} + Q_1^{-1}(2^\ell)(x \log^2(x)) + Q(x \log(x))} \\ &\ll_A \log(x) \sqrt{Q_1 \log(Q) \cdot x^2 e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)}} + \sqrt{Q_1^{-1} x^2 \log^2(x) + \log(x)} \sqrt{Q(x \log(x))}. \end{aligned}$$

Hence the contribution to (6) is bounded by the square of this:

$$\ll_A Q_1 \log(Q) \log^2(x) x^2 e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)} + Q_1^{-1} x^2 \log^2(x) + Q x \log^3(x).$$

Recalling that $Q_1 = \log^{A+1}(x)$ and $x \log^{-A}(x) \leq Q \leq x$, we see that this is

$$\ll_A \log^{A+4}(x) x^2 e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)} + Q x \log(x) + Q x \log^3(x).$$

Since $e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)} \ll_A \log^{-2A-1}(x)$, this first term is $\ll_A Q x \log^3(x)$, as required. This completes the proof of Theorem 4.

3.3. An averaged variant of Erdős' Conjecture

We now apply our variational form of the Barban–Davenport–Halberstam Theorem to prove Corollary 8.

Corollary 8. *Let $A > 0$. For all positive real numbers x and Q satisfying $x(\log(x))^{-A} \leq Q \leq x$,*

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a, q)=1}} \sum_{\substack{p_{i+1}^{a, q} \\ p_{i+1}^{a, q} \leq x}} \left(\frac{p_{i+1}^{a, q} - p_i^{a, q}}{\phi(q)} \right)^2 \ll Q x \log^3(x).$$

Proof. For each fixed a, q , we consider a partition in \mathcal{P}_x containing all the intervals of the form

$$I_i := (p_i^{a, q}, p_{i+1}^{a, q}).$$

By definition of θ and I_i , the quantity $\theta(I_i; q, a)$ equals 0 for all i . Hence,

$$\left(\theta(I_i; q, a) - \frac{|I_i|}{\phi(q)} \right)^2 = \left(\frac{p_{i+1}^{a, q} - p_i^{a, q} - 1}{\phi(q)} \right)^2.$$

We note that $(p_{i+1}^{a,q} - p_i^{a,q} - 1)^2 \gg (p_{i+1}^{a,q} - p_i^{a,q})^2$ (except for the case where $p_{i+1}^{a,q} = 3$ and $p_i^{a,q} = 2$, but this only occurs for $q = 1$ and so can be ignored). Thus, Theorem 4 implies

$$\sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a,q)=1}} \sum_{p_{i+1}^{a,q} \leq x} \left(\frac{p_{i+1}^{a,q} - p_i^{a,q}}{\phi(q)} \right)^2 \ll Qx \log^3(x). \quad \square$$

4. Another variational form of the Barban–Davenport–Halberstam Theorem

We now prove:

Theorem 5. Let $A > 0$. For all positive real numbers x and Q satisfying $x(\log(x))^{-A} \leq Q \leq x$,

$$\sum_{q \leq Q} \max_{\pi \in \mathcal{P}_x} \sum_{\substack{a \leq q \\ (a,q)=1}} \sum_{l \in \pi} \left(\theta(l; a, q) - \frac{|l|}{\phi(q)} \right)^2 \ll_A xQ \log^2(x).$$

We will need some additional notation. We let $e(x) := e^{2\pi i x}$. For any real numbers $\{a_n\}_{n=1}^N$, we define a function $S: \mathbb{T} \rightarrow \mathbb{C}$ by

$$S(\alpha) := \sum_{n=1}^N a_n e(n\alpha).$$

For $\delta > 0$, we say that points $\alpha_1, \dots, \alpha_R \in \mathbb{T}$ are δ -separated if $\|\alpha_i - \alpha_j\| \geq \delta$ for all $i \neq j$, where the $\|\cdot\|$ denotes the norm modulo 1.

We let \mathcal{P}_N denote the set of all partitions of $[N]$ into a union of disjoint intervals. We then define

$$\|S(\alpha)\|_{V^r} := \max_{\pi \in \mathcal{P}_N} \left(\sum_{l \in \pi} \left| \sum_{n \in l} a_n e(n\alpha) \right|^r \right)^{\frac{1}{r}}.$$

We note the variational Carleson Theorem [13]:

Theorem 19. For any real numbers $\{a_n\}_{n=1}^N$ and any $r > 2$,

$$\int_{\mathbb{T}} \|S(\alpha)\|_{V^r}^2 d\alpha \ll_r \sum_{n=1}^N |a_n|^2.$$

The case of $r = 2$ is addressed in the following theorem, which follows immediately from Corollary 4 in [8]:

Theorem 20. For any real numbers $\{a_n\}_{n=1}^N$,

$$\int_{\mathbb{T}} \|S(\alpha)\|_{V^2}^2 d\alpha \ll \log(N) \sum_{n=1}^N |a_n|^2.$$

We note that the $\log(N)$ factor is known to be sharp. We will first prove the following lemma, which is a variational version of the analytic large sieve inequality.

Lemma 21. For any $\delta > 0$ and for any points $\alpha_1, \dots, \alpha_R \in \mathbb{T}$ that are δ -separated,

$$\sum_{i=1}^R \|S(\alpha_i)\|_{V^r}^2 \ll_r (N + \delta^{-1} + 1) \sum_{n=1}^N |a_n|^2$$

for any $r > 2$. Also,

$$\sum_{i=1}^R \|S(\alpha_i)\|_{V^2}^2 \ll (N + \delta^{-1} + 1) \log(N) \sum_{n=1}^N |a_n|^2.$$

Proof. This proof will be a variational adaptation of the proof of Theorem 5 in [11]. By a theorem of Selberg [16], there exists an entire function $K(z)$ such that K is real-valued on \mathbb{R} , $K(x) \geq 0$ for all real x , and $K(x) \geq 1$ for all $1 \leq x \leq N$. Moreover, $K(x)$ is integrable, and $\hat{K}(0) = N - 1 + \delta^{-1}$. By a theorem of Fejér, there is another entire function $k(z)$ such that $K(x) = |k(x)|^2$ for all $x \in \mathbb{R}$, and \hat{k} (the Fourier transform of $k(x)$) has support in $(-\frac{\delta}{2}, \frac{\delta}{2})$. We note that

$$k(x) = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \hat{k}(\xi) e(x\xi) d\xi.$$

We define a function $T : \mathbb{T} \rightarrow \mathbb{C}$ by

$$T(\alpha) := \sum_{n=1}^N a_n k(n)^{-1} e(n\alpha).$$

Similarly, we define

$$\|T(\alpha)\|_{V^r} := \max_{\pi \in \mathcal{P}_N} \left(\sum_{I \in \pi} \left| \sum_{n \in I} a_n k(n)^{-1} e(n\alpha) \right|^r \right)^{\frac{1}{r}}.$$

For any α and any $r \geq 2$, we have

$$\begin{aligned} \|S(\alpha)\|_{V^r} &= \max_{\pi \in \mathcal{P}_N} \left(\sum_{I \in \pi} \left| \sum_{n \in I} a_n e(n\alpha) \right|^r \right)^{\frac{1}{r}} \\ &= \max_{\pi \in \mathcal{P}_N} \left(\sum_{I \in \pi} \left| \sum_{n \in I} a_n k(n)^{-1} e(n\alpha) \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \hat{k}(\xi) e(n\xi) d\xi \right|^r \right)^{\frac{1}{r}} \\ &= \max_{\pi \in \mathcal{P}_N} \left(\sum_{I \in \pi} \left| \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \hat{k}(\xi) \sum_{n \in I} a_n k(n)^{-1} e(n(\alpha + \xi)) d\xi \right|^r \right)^{\frac{1}{r}}. \end{aligned}$$

By Minkowski's integral inequality (see [4, Theorem 201] for example), this last quantity is

$$\begin{aligned} &\leq \max_{\pi \in \mathcal{P}_N} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \left(\sum_{l \in \pi} \left| \hat{k}(\xi) \sum_{n \in l} a_n k(n)^{-1} e(n(\alpha + \xi)) \right|^r \right)^{\frac{1}{r}} d\xi \\ &= \max_{\pi \in \mathcal{P}_N} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |\hat{k}(\xi)| \left(\sum_{l \in \pi} \left| \sum_{n \in l} a_n k(n)^{-1} e(n(\alpha + \xi)) \right|^r \right)^{\frac{1}{r}} d\xi \\ &\leq \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |\hat{k}(\xi)| \max_{\pi \in \mathcal{P}_N} \left(\sum_{l \in \pi} \left| \sum_{n \in l} a_n k(n)^{-1} e(n(\alpha + \xi)) \right|^r \right)^{\frac{1}{r}} d\xi. \end{aligned}$$

Now applying the Cauchy–Schwarz inequality, we see this is

$$\leq \left(\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |\hat{k}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \|T(\xi + \alpha)\|_{V^r}^2 d\xi \right)^{\frac{1}{2}}.$$

By the properties of k and K , we have

$$\left(\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |\hat{k}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} |k(x)|^2 dx \right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} K(x) dx \right)^{\frac{1}{2}} = (\hat{K}(0))^{\frac{1}{2}} = (N - 1 + \delta^{-1})^{\frac{1}{2}}.$$

Therefore, we have shown that

$$\sum_{i=1}^R \|S(\alpha_i)\|_{V^r}^2 \leq (N - 1 + \delta^{-1}) \sum_{i=1}^R \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \|T(\xi + \alpha_i)\|_{V^r}^2 d\xi.$$

Since the α_i 's are δ -separated, the ranges $(-\frac{\delta}{2} + \alpha_i, \frac{\delta}{2} + \alpha_i)$ are disjoint in \mathbb{T} , and hence

$$\sum_{i=1}^R \|S(\alpha_i)\|_{V^r}^2 \leq (N - 1 + \delta^{-1}) \int_{\mathbb{T}} \|T(\xi)\|_{V^r}^2 d\xi. \quad (18)$$

For $r > 2$, we may apply Theorem 19 for the real numbers $\{a_n k(n)^{-1}\}_{n=1}^N$ to conclude that the right-hand side of (18) is

$$\ll_r (N - 1 + \delta^{-1}) \sum_{n=1}^N |a_n k(n)^{-1}|^2.$$

Recalling that $\frac{1}{|k(n)|^2} = \frac{1}{K(n)}$ and $K(n) \geq 1$ for all n from 1 to N , we obtain

$$\sum_{i=1}^R \|S(\alpha_i)\|_{V^r}^2 \ll_r (N-1+\delta^{-1}) \sum_{n=1}^N |a_n|^2$$

for all $r > 2$, as required.

For $r = 2$, we may apply Theorem 20 for the real numbers $\{a_n k(n)^{-1}\}_{n=1}^N$ to conclude that the right-hand side of (18) is

$$\ll (N-1+\delta^{-1}) \log(N) \sum_{n=1}^N |a_n k(n)^{-1}|^2 \ll (N-1+\delta^{-1}) \log(N) \sum_{n=1}^N |a_n|^2. \quad \square$$

We next prove the following lemma:

Lemma 22. For any real numbers $\{a_n\}_{n=1}^N$,

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \max_{\pi \in \mathcal{P}_N} \sum_{\chi \bmod q}^* \sum_{l \in \pi} |T(l, \chi)|^2 \ll (N+Q^2) \log(N) \sum_{n=1}^N |a_n|^2.$$

Proof. This proof is adapted from the proof of Theorem 4 in [2]. For a character χ modulo q , we define the complex value $\tau(\chi)$ by

$$\tau(\chi) := \sum_{a \leq q} \chi(a) e\left(\frac{a}{q}\right).$$

We have

$$\chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_{a \leq q} \bar{\chi}(a) e\left(\frac{an}{q}\right)$$

for all n for all primitive χ (see [2, Chapter 9]). Therefore, for any interval I and any primitive χ modulo q ,

$$T(I, \chi) = \sum_{n \in I} a_n \chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_{a \leq q} \bar{\chi}(a) \sum_{n \in I} a_n e\left(\frac{an}{q}\right).$$

We then note that when χ is primitive, $|\tau(\chi)| = q^{\frac{1}{2}}$ (see [2, p. 66]). This yields (for any partition π):

$$\sum_{l \in \pi} |T(l, \chi)|^2 = \frac{1}{q} \sum_{l \in \pi} \left| \sum_{a \leq q} \bar{\chi}(a) \sum_{n \in l} a_n e\left(\frac{an}{q}\right) \right|^2.$$

Now summing over primitive characters, we have

$$\sum_{\chi \bmod q}^* \sum_{l \in \pi} |T(\chi, l)|^2 = \frac{1}{q} \sum_{l \in \pi} \sum_{\chi \bmod q}^* \left| \sum_{a \leq q} \bar{\chi}(a) \sum_{n \in l} a_n e\left(\frac{an}{q}\right) \right|^2.$$

This quantity can only increase if we sum over all characters modulo q , so this is

$$\leq \frac{1}{q} \sum_{l \in \pi} \sum_{a \leq q} \sum_{b \leq q} \left(\sum_{n \in l} a_n e\left(\frac{an}{q}\right) \right) \left(\sum_{n \in l} a_n \bar{e}\left(\frac{bn}{q}\right) \right) \sum_{\chi \bmod q} \bar{\chi}(a) \chi(b). \quad (19)$$

Here, \bar{e} denotes conjugation. We note that this innermost sum over characters is equal to 0 unless $a = b$ and a is coprime to q . In this case, it is equal to $\phi(q)$. Plugging this into (19), we have

$$= \frac{\phi(q)}{q} \sum_{l \in \pi} \sum_{\substack{a \leq q \\ (a, q) = 1}} \left| \sum_{n \in l} a_n e\left(\frac{an}{q}\right) \right|^2.$$

Recalling that

$$\|S(\alpha)\|_{V^2}^2 := \max_{\pi \in \mathcal{P}_N} \sum_{l \in \pi} \left| \sum_{n \in l} a_n e(n\alpha) \right|^2,$$

we conclude that

$$\begin{aligned} \sum_{q \leq Q} \frac{q}{\phi(q)} \max_{\pi \in \mathcal{P}_N} \sum_{\chi \bmod q}^* \sum_{l \in \pi} |T(l, \chi)|^2 &\ll \sum_{q \leq Q} \max_{\pi \in \mathcal{P}_N} \sum_{l \in \pi} \sum_{\substack{a \leq q \\ (a, q) = 1}} \left| \sum_{n \in l} a_n e\left(\frac{an}{q}\right) \right|^2 \\ &\leq \sum_{q \leq Q} \sum_{\substack{a \leq q \\ (a, q) = 1}} \|S(a/q)\|_{V^2}^2. \end{aligned}$$

Here we have used the fact that moving the maximum inside the sum over a 's coprime to q can only make the quantity larger. The points $\frac{a}{q}$ as q ranges from 1 to Q and a ranges over values coprime to q are $\frac{1}{Q^2}$ -separated as points in \mathbb{T} . Thus, applying Lemma 21, this is $\ll (N + Q^2) \log(N) \sum_{n=1}^N |a_n|^2$. \square

We are now equipped to prove Theorem 5. We recall Lemma 17, which states that

$$\theta(I; q, a) - \frac{|I|}{\phi(q)} = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \theta'(I, \chi).$$

We then have

$$\sum_{q \leq Q} \max_{\pi \in \mathcal{P}_x} \sum_{\substack{a \leq q \\ (a, q) = 1}} \sum_{l \in \pi} \left(\theta(I; q, a) - \frac{|I|}{\phi(q)} \right)^2 = \sum_{q \leq Q} \frac{1}{\phi(q)^2} \max_{\pi \in \mathcal{P}_x} \sum_{\substack{a \leq q \\ (a, q) = 1}} \sum_{l \in \pi} \left| \sum_{\chi \bmod q} \bar{\chi}(a) \theta'(I, \chi) \right|^2.$$

By expanding the square and rearranging the sums inside the maximum, this is

$$= \sum_{q \leq Q} \frac{1}{\phi(q)^2} \max_{\pi \in \mathcal{P}_x} \sum_{l \in \pi} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} \theta'(I, \chi_1) \overline{\theta'(I, \chi_2)} \sum_{\substack{a \leq q \\ (a, q) = 1}} \bar{\chi}_1(a) \chi_2(a).$$

This innermost sum over a 's coprime to q is equal to $\phi(q)$ whenever $\chi_1 = \chi_2$, and is equal to 0 otherwise. Hence this quantity is

$$= \sum_{q \leq Q} \frac{1}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \sum_{l \in \pi} \sum_{\chi \bmod q} |\theta'(l, \chi)|^2. \quad (20)$$

Each character $\chi \bmod q$ is induced by some primitive character $\chi_1 \bmod q_1$, where q_1 divides q . Applying the triangle inequality, we see that (20) is

$$\begin{aligned} &\ll \sum_{q \leq Q} \frac{1}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \sum_{l \in \pi} \sum_{\chi \bmod q} |\theta'(l, \chi_1)|^2 \\ &\quad + \sum_{q \leq Q} \frac{1}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \sum_{l \in \pi} \sum_{\chi \bmod q} |\theta'(l, \chi) - \theta'(l, \chi_1)|^2. \end{aligned} \quad (21)$$

We consider the second quantity in (21). This is

$$\leq \sum_{q \leq Q} \frac{1}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \left(\sum_{l \in \pi} \sum_{\chi \bmod q} |\theta'(l, \chi) - \theta'(l, \chi_1)| \right)^2.$$

Recalling the definitions of $\theta'(l, \chi)$ and $\theta'(l, \chi_1)$, we note that

$$|\theta'(l, \chi) - \theta'(l, \chi_1)| \leq \sum_{\substack{p \in l \\ p|q}} \log p.$$

Thus, the second quantity in (21) is

$$\leq \sum_{q \leq Q} \frac{1}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \left(\sum_{l \in \pi} \sum_{\chi \bmod q} \sum_{\substack{p \in l \\ p|q}} \log(p) \right)^2 = \sum_{q \leq Q} \phi(q) \left(\sum_{\substack{p \leq x \\ p|q}} \log(p) \right)^2.$$

Since $\sum_{p|q} \log(p) \leq \log(q)$, this is

$$\leq \sum_{q \leq Q} \phi(q) \log^2(q) \ll Q^2 \log^2(Q) \leq xQ \log^2(x),$$

for $Q \leq x$.

It now suffices to bound the first quantity in (21). Every primitive character $\chi_1 \bmod q_1$ induces characters modulo q for every q that is a multiple of q_1 . Also, the set of primitive characters χ_1 inducing characters modulo q can be divided into primitive characters modulo each divisor of q . By applying the triangle inequality and maximizing separately for each divisor of q , we see that for each q :

$$\max_{\pi \in \mathcal{P}_x} \sum_{l \in \pi} \sum_{\chi \bmod q} |\theta'(l, \chi_1)|^2 \leq \sum_{q_1|q} \max_{\pi \in \mathcal{P}_x} \sum_{l \in \pi} \sum_{\chi_1 \bmod q_1}^* |\theta'(l, \chi_1)|^2.$$

Now summing over q and accounting for the multiple occurrences of each q_1 , we have

$$\sum_{q \leq Q} \frac{1}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \sum_{\pi \in I} \sum_{\chi \bmod q} |\theta'(I, \chi_1)|^2 \leq \sum_{q \leq Q} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \sum_{\chi \bmod q}^* |\theta'(I, \chi)|^2 \left(\sum_{j \leq \frac{Q}{q}} \frac{1}{\phi(jq)} \right).$$

As previously noted, this final sum over j is $\ll \phi(q)^{-1} \log(2Q/q)$. Hence the above quantity is

$$\ll \sum_{q \leq Q} \frac{\log(2Q/q)}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \sum_{\chi \bmod q}^* |\theta'(I, \chi)|^2.$$

As in the proof of Theorem 4, we break this sum over q into smaller ranges. For any $1 \leq U \leq Q$, we have

$$\begin{aligned} & \sum_{U < q \leq 2U} \frac{\log(2Q/q)}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \sum_{\chi \bmod q}^* |\theta'(I, \chi)|^2 \\ & \ll U^{-1} \log(2Q/U) \sum_{q \leq 2U} \frac{q}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \sum_{\chi \bmod q}^* |\theta(I, \chi)|^2 \end{aligned} \quad (22)$$

(note that the change of notation from θ' to θ does not change any values). We may now apply Lemma 22 with $a_p = \log(p)$ for all primes p and $a_n = 0$ otherwise. We conclude that the right-hand side of (22) is

$$\ll U^{-1} \log(2Q/U) (x + U^2) \log(x) \left(\sum_{p \leq x} \log^2(p) \right) \ll U^{-1} \log(2Q/U) (x + U^2) x \log^2(x).$$

We define $Q_1 = \log^{A+1}(x)$. Setting $U = Q 2^{-j}$ and summing over j from 0 to $J := \lceil \log(Q/Q_1) \rceil$, we see that

$$\sum_{Q_1 < q \leq Q} \frac{\log(2Q/q)}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \sum_{\chi \bmod q}^* |\theta'(I, \chi)|^2 \ll Q^{-1} x \log^2(x) \sum_{j=0}^J 2^j (j+1) (x + Q^2 2^{-2j}).$$

Since $\sum_{j=0}^{\infty} (j+1) 2^{-j}$ converges and $\sum_{j=0}^J (j+1) 2^j \ll J 2^J$, this quantity is $\ll Q_1^{-1} x^2 \log^2(x) \log(Q) + xQ \log^2(x)$. Recalling that $Q_1 = \log^{A+1}(x)$ and $x \log^{-A}(x) \leq Q \leq x$, we see this is $\ll xQ \log^2(x)$ as required.

It only remains to bound the contribution from the values of $q \leq Q_1$. We observe that

$$\sum_{q \leq Q_1} \frac{\log(2Q/q)}{\phi(q)} \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} \sum_{\chi \bmod q}^* |\theta'(I, \chi)|^2 \ll \log(Q) \sum_{q \leq Q_1} \frac{1}{\phi(q)} \sum_{\chi \bmod q}^* \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} |\theta'(I, \chi)|^2.$$

Now, every primitive character modulo q for $q > 1$ is non-principal. For the principal character χ_0 modulo 1, the value of $\theta'(I, \chi_0)$ is 0 for any I , so we may rewrite our quantity as

$$\log(Q) \sum_{1 < q \leq Q_1} \frac{1}{\phi(q)} \sum_{\chi \bmod q}^* \max_{\pi \in \mathcal{P}_x} \sum_{I \in \pi} |\theta(I, \chi)|^2.$$

Applying Lemma 14 for the constant $A + 1$, we see this is

$$\ll_A \log(Q) \sum_{1 < q \leq Q_1} \frac{1}{\phi(q)} \sum_{\chi \bmod q}^* x^2 e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)},$$

for some positive constant \tilde{c}_A depending only on A . Since there are $\phi(q)$ characters modulo q (and hence at most $\phi(q)$ primitive ones), this is $\ll_A Q_1 \log(Q) x^2 e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)}$. Recalling that $Q \leq x$ and $Q_1 = \log^{A+1}(x)$ and noting that $e^{-\tilde{c}_A \log^{\frac{1}{2}}(x)} \ll_A \log^{-2A}(x)$, we see this $\ll_A x Q \log^2(x)$. This completes the proof of Theorem 5.

5. A variational form of the large sieve inequality

We now prove another variational form of Theorem 11. This will refine an estimate of Uchiyama stated below. The techniques are similar to those used above. We let $\mathcal{P}_{M,N}$ denote the set of partitions of $[M + 1, M + N]$.

Lemma 23. For all positive integers Q, M, N and complex numbers $\{a_n\}_{n=M+1}^{M+N}$,

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* \max_{\pi \in \mathcal{P}_{M,N}} \sum_{I \in \pi} |T(\chi, I)|^2 \ll \log^2(N)(N + Q^2) \sum_{n=M+1}^{N+M} |a_n|^2. \quad (23)$$

Proof. Without loss of generality, we assume that $N = 2^k$ for some k (rounding N up to the nearest power of 2 can be absorbed by the implicit constant). The interval from $M + 1$ to $M + N$ can then be decomposed into dyadic intervals of the form $I_{c,\ell} := (M + (c - 1)2^\ell, M + c2^\ell]$ for each ℓ from 0 to k , where c ranges from 1 to $2^{k-\ell}$. If we fix ℓ and let c vary, we refer to the resulting set of intervals as the dyadic intervals on level ℓ .

For a fixed q and primitive character $\chi \bmod q$, we consider a maximizing partition π^* in $\mathcal{P}_{M,N}$. By Lemma 16, every interval $I \in \pi^*$ can be decomposed as a union of $O(\log N)$ dyadic intervals of the form $I_{c,\ell}$ for varying c and ℓ values, such that there are at most 2 intervals included on each level. We let $D(I)$ denote the set of dyadic intervals in the decomposition of I . For each ℓ , $D_\ell(I)$ denotes the subset of intervals in $D(I)$ on level ℓ .

We can now express the square root of the left-hand side of (23) as:

$$\sqrt{\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* \sum_{I \in \pi^*} \left| \sum_{J \in D(I)} T(\chi, J) \right|^2}.$$

We can alternatively express this as:

$$\sqrt{\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* \sum_{I \in \pi^*} \left| \sum_{\ell=0}^k \sum_{J \in D_\ell(I)} T(\chi, J) \right|^2}.$$

Applying the triangle inequality for the ℓ^2 norm, this quantity is

$$\leq \sum_{\ell=0}^k \sqrt{\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* \sum_{I \in \pi^*} \left| \sum_{J \in D_\ell(I)} T(\chi, J) \right|^2}. \quad (24)$$

For a fixed ℓ , we consider the quantity

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* \sum_{I \in \pi^*} \left| \sum_{J \in D_\ell(I)} T(\chi, J) \right|^2. \quad (25)$$

We note that the innermost sum contains at most two intervals J , since the dyadic decomposition of each I contains at most two intervals on level ℓ . Noting that $|a + b|^2 \ll |a|^2 + |b|^2$ holds for all complex numbers a and b and that each dyadic interval on level ℓ can occur in the decomposition of at most one $I \in \pi^*$, the quantity in (25) is

$$\ll \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* \sum_{c=1}^{2^{k-\ell}} |T(\chi, I_{c,\ell})|^2.$$

Now the innermost sum is simply over the set of dyadic intervals on level ℓ . Reordering the finite sums, this is

$$= \sum_{c=1}^{2^{k-\ell}} \sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* |T(\chi, I_{c,\ell})|^2.$$

For each c , we can apply Theorem 11 to obtain:

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* |T(\chi, I_{c,\ell})|^2 \ll (2^\ell + Q^2) \sum_{n \in I_{c,\ell}} |a_n|^2.$$

Thus, the quantity in (25) is $\ll (2^\ell + Q^2) \sum_{n=M+1}^{M+2^k} |a_n|^2$.

Substituting this back into (24), we see that the square root of the left-hand side of (23) is

$$\ll \sum_{\ell=0}^k \sqrt{(2^\ell + Q^2) \sum_{n=M+1}^{M+2^k} |a_n|^2}.$$

Hence, the left-hand side of (23) is

$$\ll \left(\sum_{\ell=0}^k \sqrt{2^\ell + Q^2} \right)^2 \sum_{n=M+1}^{M+2^k} |a_n|^2.$$

Recalling that $2^k = N$ and loosely bounding $2^\ell + Q^2 \leq 2^k + Q^2$ for all ℓ , we see this is

$$\ll (k\sqrt{2^k + Q^2})^2 \sum_{n=M+1}^{M+2^k} |a_n|^2 = k^2(2^k + Q^2) \sum_{n=M+1}^{M+2^k} |a_n|^2 \ll \log^2(N)(N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2. \quad \square$$

We note that this refines Uchiyama's maximal large sieve inequality [15], which states:

Lemma 24. For all positive integers Q, M, N and complex numbers $\{a_n\}_{n=M+1}^{M+N}$,

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\chi \bmod q}^* \max_{I \subseteq [N]} |T(\chi, I)|^2 \ll \log^2(N)(N + Q^2) \sum_{n=M+1}^{N+M} |a_n|^2. \quad (26)$$

Montgomery [10] has asked if the $\log^2(N)$ can be removed. We do not have an answer to this question (though we obtain a lower bound on a related quantity in [9]). We note that the $\log^2(N)$ cannot be completely removed in our variational refinement.

To see this, we use the lemma below, which follows easily from Lemma 22 in [8] and the Cauchy–Schwarz inequality:

Lemma 25. Let c_1, \dots, c_N denote complex numbers $|c_i| \geq \delta$, for some $\delta > 0$. Let X_1, \dots, X_N denote independent Gaussian random variables each with mean 0 and variance 1. Then

$$\mathbb{E} \left[\sup_{\pi \in \mathcal{P}_N} \sum_{I \in \pi} \left| \sum_{n \in I} c_n X_n \right|^2 \right] \gg \delta^2 N \log \log(N).$$

Strictly speaking, the lemma in [8] is stated for real c_1, \dots, c_N , but it can easily be deduced for complex numbers by splitting into real and imaginary parts. Now we consider (23) with interval $[N]$ and $a_n = X_n$ for each $n \in [N]$. To apply Lemma 25 for each q and each character modulo q , we consider only the indices n such that n is coprime to q . On these values, the character will be nonzero. We let $C(q)$ denote the number of these indices for each q . Then, from Lemma 25, the expectation of the left-hand side can be estimated by

$$\gg \sum_{q \leq Q} q \cdot C(q) \cdot \log \log(C(q)). \quad (27)$$

Now, we note that $\sum_{q \leq Q} C(q)$ is equal to the number of pairs (n, q) such that n and q are coprime. This is also equal to $\sum_{n=1}^N Q \frac{\phi(n)}{n}$. By Theorem 330 in [5], $\sum_{n=1}^N \phi(n) = \frac{3}{\pi^2} N^2 + O(N \log(N))$, which implies $\sum_{n=1}^N \frac{\phi(n)}{n} \gg N$. Thus, $\sum_{q \leq Q} C(q) \gg QN$. It follows that there exist positive constants ϵ, δ such that $C(q) \geq \epsilon N$ for at least δQ values of q . Hence, the quantity in (27) is $\gg Q^2 N \log \log(N)$, while the expectation of the sum of squares of the $a_n = X_n$ will be $\ll N$. Thus, at least when $N \ll Q^2$, one needs at least a factor of $\log \log(N)$ in (23). Refining the gap between $\log \log(N)$ and $\log^2(N)$ would be interesting.

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